An $\alpha$-Cut Approach for Fuzzy Product and Its Use in Computing Solutions of Fully Fuzzy Linear Systems

R. Hassanzadeh and I. Mahdavi
Department of Industrial Engineering
Mazandaran University of Science and Technology
Babol, Iran

N. Mahdavi-Amiri
Faculty of Mathematical Sciences
Sharif University of Technology
Tehran, Iran

A. Tajdin
Department of Industrial Engineering
Mazandaran University of Science and Technology
Babol, Iran

Abstract

We propose an approach for computing the product of various fuzzy numbers using $\alpha$-cuts. A regression model is used to obtain the membership function of the product. Then, we make use of the approach to compute solutions of fully fuzzy linear systems. We also show how to compute solutions of fully fuzzy linear systems with various fuzzy variables. Examples are worked out to illustrate the approach.

Keywords
Fuzzy numbers; $\alpha$-cut; regression model; fuzzy product; fully fuzzy linear system (FFLS)

1. Introduction

Over the years, a number of researchers have focused on computing various approximations of product of fuzzy numbers (see, e.g., Ban 2008 and 2009, Mrówka 2005, Nasibov and Peker 2008, Yeh 2009, Ghanbari et al. 2010, Ezzati to appear). Here, we propose a new approach for computing fuzzy product of fuzzy numbers using $\alpha$-cuts. The fuzzy numbers may be of the same type or mixed. Linear systems arise from many areas of science and engineering. Moore (1979) pointed out that exact numerical data might be unrealistic, but uncertain data can consider more aspects of a real world problem. A popular way to describe uncertain data is using fuzzy data. Fuzzy linear systems under various assumption have been studied. Buckley et al. (1991 and 2002) studied square fuzzy linear systems, $A\vec{x} = \vec{b}$, where the components of $A$ and $\vec{b}$ are fuzzy numbers. Since an algebraic solution does not exist, they proposed three other types of solutions (based on $\alpha$-cuts, extension principal and interval arithmetic). Vorman et al. (2007) improved Buckley and Qu’s method (1991) by obtaining the necessary parametric functions needed for the solution of the systems corresponding to the bounds of the considered $\alpha$-level. Muzzioli and Reynaerts (2006) provided a generalization of the vector solution proposed by Buckley and Qu (1991) to the fuzzy system $A_1\vec{x} + \vec{b}_1 = A_2\vec{x} + \vec{b}_2$, where the components of $A_1, A_2, \vec{b}_1$ and $\vec{b}_2$ are triangular fuzzy numbers and $A_1$ and $A_2$ are square matrices. They applied their method to solve and analyze a particular fuzzy linear system having financial applications (Muzzioli and Reynaerts 2007).

A general model for solving a fuzzy linear system whose coefficient matrix is crisp and the right-hand side column is an arbitrary fuzzy vector was first proposed by Friedman et al. (1998). Friedman and his colleagues used the embedding method and replaced the original fuzzy linear system by a crisp linear system. Following their work, Dehghan and Hashemi (2006)...
proposed some iterative methods. Ghanbari et al. (2010) have recently showed shortcomings of the approach proposed by Friedman et al. (1998) and proposed a new approach for computing the solutions based on a least squares model.

As noticed by Rao and Chen (1998) and Muzzioli and Reynaerts (2006), although several investigations have been reported in the literature for the solution of fuzzy systems (Buckley and Qu 1991, Zhao and Govind 1991, Kawaguchi and Da-Te 1993), very few methods are in fact practical. The main reason for this drawback lies in the structure of the problem. To be more precise, as pointed out by Kreinovich et al. (1998), finding a solution of a system with interval coefficients is NP-hard. Here, we discuss the case in which all parameters in a fuzzy linear system are fuzzy numbers, calling it a Fully Fuzzy Linear System (FFLS). We intend to propose an approach for solving \( \tilde{A} \odot \tilde{x} = \tilde{b} \), where \( \tilde{A} \) is a fuzzy matrix and \( \tilde{x} \) and \( \tilde{b} \) are fuzzy vectors. Rao and Chen (1998) gave a computational method to solve this problem, based on their proposed cuts. But a computational method using Zadeh’s extension principle is not at hand (see, e.g., Buckley and Qu 1991). Since there is no analytic formula for some arithmetic operators, Dubois and Prade (1980) proposed some approximations for arithmetic operators lacking analytic formulas on a special and useful type of fuzzy numbers, that are used by other researchers Wagenknecht et al. 2001, Sakawa 1973, Giachetti and Young 1997.

In Section 2, basic definitions are given. Section 3 describes \( \alpha \)-cuts. The design of fuzzy product operator on fuzzy numbers is described in Section 4. Solutions of fully fuzzy linear systems are described in Section 5. We conclude in Section 6.

2. Definitions
To arrive at the product of various types of fuzzy numbers, we start with basic definitions of some well-known fuzzy numbers.

**Definition 1.** An LR fuzzy number is represented by \( \tilde{A} = (m, a, b)_{LR} \), with the membership function, \( \mu_{\tilde{a}}(x) \), defined by

\[
\mu_{\tilde{a}}(x) = \begin{cases} 
\frac{m-x}{a} & x \leq m \\
\frac{x-m}{b} & x \geq m,
\end{cases}
\]

where, \( L \) and \( R \) are non-increasing functions from \( \mathbb{R}^+ \) to \([0,1]\), \( L(0)=R(0)=1 \), \( m \) is the center, \( a \) is the left spread and \( b \) is the right spread [18].

Note that if \( L(x) = R(x) = 1-x \) with \( 0<x<1 \), then \( x \) is a triangular fuzzy number and is represented by the triplet \( \tilde{a} = (m, a, b) \), with the membership function \( \mu_{\tilde{a}}(x) \), defined by

\[
\mu_{\tilde{a}}(x) = \begin{cases} 
1- \frac{m-x}{a} & x \leq m \\
1- \frac{x-m}{b} & x \geq m.
\end{cases}
\]

**Definition 2.** A trapezoidal fuzzy number \( \tilde{a} \) is shown by \( \tilde{a} = (a_1, a_2, a_3, a_4) \), with the membership function as follows:

\[
\mu_{\tilde{a}}(x) = \begin{cases} 
0 & x \leq a_1 \\
\frac{x-a_1}{a_2-a_1} & a_1 \leq x \leq a_2 \\
1 & a_2 \leq x \leq a_3 \\
\frac{a_4-x}{a_4-a_3} & a_3 \leq x \leq a_4 \\
0 & a_4 \leq x.
\end{cases}
\]

It is apparent that a triangular fuzzy number is a special trapezoidal fuzzy number with \( a_2 = a_3 \).

**Definition 3.** If \( L(x)=R(x)= e^{-x^2} \), with \( x \in \mathbb{R} \), then \( x \) is a normal fuzzy number that is shown by \( (m, \sigma) \) and the corresponding membership function is defined to be:

\[
\mu_{\tilde{a}}(x) = e^{-\frac{(x-m)^2}{\sigma}}, \quad x \in \mathbb{R},
\]

where, \( m \) is the mean and \( \sigma \) is the standard deviation.
**Definition 4.** The $\alpha$-cut and strong $\alpha$-cut for a fuzzy set $\tilde{A}$ are shown by $\tilde{A}_\alpha$ and $\tilde{A}_\alpha^+$, respectively, and for $\alpha \in [0,1]$ are defined to be:

$$\tilde{A}_\alpha = \{ x \mid \mu_\tilde{A}(x) \geq \alpha, x \in X \},$$

$$\tilde{A}_\alpha^+ = \{ x \mid \mu_\tilde{A}(x) > \alpha, x \in X \},$$

where, $X$ is the universal set.

Upper and lower bounds for any $\alpha$-cut $\tilde{A}_\alpha$ are shown by $\sup \tilde{A}_\alpha$ and $\inf \tilde{A}_\alpha$, respectively. Here, we assume that the upper and lower bounds of $\alpha$-cuts are finite values and for simplicity denote $\sup \tilde{A}_\alpha$ by $\tilde{A}_\alpha^+$ and $\inf \tilde{A}_\alpha$ by $\tilde{A}_\alpha^-$.

**Definition 5.** Let $\tilde{A} = (\tilde{a}_{ij})$ and $\tilde{B} = (\tilde{b}_{ij})$ be $m \times n$ and $n \times p$ fuzzy matrices respectively. We define $\tilde{A} \otimes \tilde{B} = \tilde{C}$ to be an $m \times p$ matrix, where,

$$\tilde{c}_{ij} = \sum_{k=1}^{n} (\tilde{a}_{ik} \otimes \tilde{b}_{kj}).$$

**Definition 6.** Consider the $n \times n$ linear system of equations:

$$\begin{align*}
(\tilde{a}_{11} \otimes \tilde{x}_1) + (\tilde{a}_{12} \otimes \tilde{x}_2) + \cdots + (\tilde{a}_{1n} \otimes \tilde{x}_n) &= \tilde{b}_1, \\
(\tilde{a}_{21} \otimes \tilde{x}_1) + (\tilde{a}_{22} \otimes \tilde{x}_2) + \cdots + (\tilde{a}_{2n} \otimes \tilde{x}_n) &= \tilde{b}_2, \\
& \vdots \\
(\tilde{a}_{n1} \otimes \tilde{x}_1) + (\tilde{a}_{n2} \otimes \tilde{x}_2) + \cdots + (\tilde{a}_{nn} \otimes \tilde{x}_n) &= \tilde{b}_n.
\end{align*}$$

In matrix form, the above equations are shown by $\tilde{A} \otimes \tilde{x} = \tilde{b}$, or simply $\tilde{A} \tilde{x} = \tilde{b}$, where the coefficient matrix $\tilde{A} = (\tilde{a}_{ij})$, $1 \leq i, j \leq n$ is an $n \times n$ fuzzy matrix and $\tilde{x}$ and $\tilde{b}$ are $n \times 1$ fuzzy vectors. This system is called a fully fuzzy linear system (FFLS).

### 3. Computing $\alpha$-Cuts

One approach for multiplication of fuzzy numbers is to employ $\alpha$-cuts. Here, using our recently developed approach for addition of mixed fuzzy numbers in Tajdin (2010), we describe a methodology to multiply various fuzzy numbers using $\alpha$-cuts.

For an $LR$ fuzzy number with $L$ and $R$ invertible functions, the $\alpha$-cut is characterized by (see Fig. 1):

$$\alpha = L \left( \frac{m-x}{a} \right) \Rightarrow \frac{m-x}{a} = L^{-1}(\alpha) \Rightarrow \tilde{A}_\alpha^L = x = m - aL^{-1}(\alpha),$$

$$\alpha = R \left( \frac{x-m}{b} \right) \Rightarrow \frac{x-m}{b} = R^{-1}(\alpha) \Rightarrow \tilde{A}_\alpha^R = x = m + bR^{-1}(\alpha).$$

Thus, $\tilde{A}_\alpha = [\tilde{A}_\alpha^L, \tilde{A}_\alpha^R]$ is the corresponding $\alpha$-cut.

![Figure 1. An example of an $\alpha$-cut](image-url)

Next, for specific $L$ and $R$ functions, the following cases are discussed.
3.1. α-cuts for trapezoidal fuzzy numbers

Let \( \tilde{a} = (a_1, a_2, a_3, a_4) \) be a trapezoidal fuzzy number. An \( \alpha \)-cut for \( \tilde{a} \), \( \tilde{a}_\alpha \), is computed as:

\[
\alpha = \frac{x - a_1}{a_2 - a_1} \Rightarrow \tilde{a}^L_\alpha = x = (a_2 - a_1)\alpha + a_1, \\
\alpha = \frac{a_4 - x}{a_4 - a_3} \Rightarrow \tilde{a}^R_\alpha = x = a_4 - (a_4 - a_3)\alpha,
\]

where, \( \tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^R_\alpha] \) is the corresponding \( \alpha \)-cut (see Fig. 2). The \( \alpha \)-cuts for triangular fuzzy numbers are obtained by using the above equations considering \( a_2 = a_3 \).

![Figure 2. A trapezoidal fuzzy number with an α-cut](image)

3.2. α-cuts for normal fuzzy numbers

If \( \tilde{a} = (m, \sigma) \) is a normal fuzzy number, then \( \tilde{a}_\alpha \), \( 0 < \alpha \leq 1 \), is computed as:

\[
\alpha = e^{-\frac{(m-x)^2}{\sigma^2}} \Rightarrow \sqrt{-\ln(\alpha)} = \frac{m-x}{\sigma} \Rightarrow \tilde{a}^L_\alpha = x = m - \sigma\sqrt{-\ln(\alpha)} \\
\alpha = e^{-\frac{(x-m)^2}{\sigma^2}} \Rightarrow \sqrt{-\ln(\alpha)} = \frac{x-m}{\sigma} \Rightarrow \tilde{a}^R_\alpha = x = m + \sigma\sqrt{-\ln(\alpha)},
\]

where, \( \tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^R_\alpha] \), with \( 0 < \alpha \leq 1 \), is the corresponding \( \alpha \)-cut.

4. Fuzzy Product Operator

We now propose an approach for the product of mixed fuzzy numbers using \( \alpha \)-cuts. Divide the interval \([0,1]\) into \( n \) equal subintervals using \( n+1 \) equidistant points \( \alpha_0 = 0, \alpha_i = \alpha_{i-1} + \Delta \alpha, \) \( i = 1,...,n \), with \( \Delta \alpha = \frac{1}{n}, \) \( i = 1,...,n \). For normal fuzzy numbers \( x \in (-\infty, +\infty) \), it is improper to let \( \alpha \) be equal to zero. Therefore, for this case we consider \( \alpha \in (0,1] \).

Assume two fuzzy numbers \( \tilde{a} \) and \( \tilde{b} \) with corresponding \( \alpha \)-cuts \( \tilde{a}_\alpha = [\tilde{a}^-_\alpha, \tilde{a}^+_\alpha] \) and \( \tilde{b}_\alpha = [\tilde{b}^-_\alpha, \tilde{b}^+_\alpha] \). The product of the \( \alpha \)-cuts are:

\[
\tilde{a}_\alpha \times \tilde{b}_\alpha = [L, R],
\]

where,

\[
L = \min (\tilde{a}^-_\alpha \times \tilde{b}^-_\alpha, \tilde{a}^-_\alpha \times \tilde{b}^+_\alpha, \tilde{a}^+_\alpha \times \tilde{b}^-_\alpha, \tilde{a}^+_\alpha \times \tilde{b}^+_\alpha, \tilde{a}^-_\alpha \times \tilde{b}^-_\alpha), \\
R = \max (\tilde{a}^-_\alpha \times \tilde{b}^-_\alpha, \tilde{a}^-_\alpha \times \tilde{b}^+_\alpha, \tilde{a}^+_\alpha \times \tilde{b}^-_\alpha, \tilde{a}^+_\alpha \times \tilde{b}^+_\alpha).
\]

(3)
Next, we show how to add up a trapezoidal fuzzy number together with a normal one. A numerical approach is given to approximate the sum and its corresponding membership function.

**Example 1:** Consider the following two trapezoidal fuzzy numbers:

\[
\tilde{a} = (1.5, 6, 9), \quad \tilde{b} = (2, 3, 5, 8).
\]

The diagram of their product, \(\tilde{a} \times \tilde{b}\), using equation (3), for 100 \(\alpha\) cuts, \(i = 1, \ldots, 100\), is shown in figure 3. As seen in the diagram, the result is not exactly trapezoidal.

![Figure 3. The product of fuzzy numbers using \(\alpha\)-cuts](image1)

Example 2: consider the following two normal fuzzy numbers:

\[
\tilde{a} = (40, 4), \quad \tilde{b} = (50, 9).
\]

The diagram of their product, \(\tilde{a} \times \tilde{b}\), using (3), for 100 \(\alpha\)-cuts, \(i = 1, \ldots, 100\), is shown in figure 4. As seen in the diagram, the result appears to be normal.

In general, the membership function for \(\tilde{a} \times \tilde{b}\) can be computed as follows:

Consider two nonnegative fuzzy numbers \(\tilde{a} = (a_1, a_2, a_3, a_4)\) and \(\tilde{b} = (b_1, b_2, b_3, b_4)\). The end points of the \(\alpha\)-cuts for them are:

\[
\begin{align*}
\tilde{a}_a^L &= x = (a_2 - a_1)\alpha + a_1, \\
\tilde{a}_a^R &= x = a_4 - (a_4 - a_3)\alpha \\
\tilde{b}_a^L &= x = (b_2 - b_1)\alpha + b_1, \\
\tilde{b}_a^R &= x = b_4 - (b_4 - b_3)\alpha.
\end{align*}
\]

Thus, we have

\[
\tilde{c}_a^L = \tilde{a}_a^L \times \tilde{b}_a^L = (a_2 - a_1)\alpha + a_1 \times (b_2 - b_1)\alpha + b_1 = A\alpha^2 + B\alpha + C,
\]

where, \(A = (a_2 - a_1)(b_2 - b_1)\), \(B = b_1(a_2 - a_1) + a_1(b_2 - b_1)\) and \(C = a_1b_1\).

Now, using (4), we have

\[
\tilde{c}_a^L = \tilde{a}_a^L \times \tilde{b}_a^L = x = A(\alpha^2 + B\alpha + \frac{B^2}{4A^2} - \frac{B^2}{4A^2}) + C \quad \Rightarrow \quad \alpha = \frac{\sqrt{4Ax - 4AC + B^2} - B}{2A}.
\]

Also, for the right side of the \(\alpha\)-cuts, we have

\[
\tilde{c}_a^R = \tilde{a}_a^R \times \tilde{b}_a^R = \alpha' A^2 + B'\alpha + C',
\]

where, \(A' = (a_4 - a_3)(b_4 - b_3)\), \(B' = b_4(a_3 - a_4) + a_4(b_4 - b_3)\) and \(C' = a_3b_4\).

Using (6), we have

\[
\tilde{c}_a^R = \tilde{a}_a^R \times \tilde{b}_a^R = x = A'(\alpha^2 + B'\alpha + \frac{B'^2}{4A'^2} - \frac{B'^2}{4A'^2}) + C' \quad \Rightarrow \quad \alpha = \frac{\sqrt{4Ax - 4AC' + B'^2} - B'}{2A'}.
\]

Therefore, using (5) and (7), the membership function for the product of the two nonnegative fuzzy numbers is:
The membership function for the product of the two nonnegative normal fuzzy numbers is:

\[
\mu_i(x) = \begin{cases} 
\frac{\sqrt{4Ax - 4AC + B^2} - B}{2A} & x \leq a_i b_i \\
1 & a_i b_1 < x < a_i b_2 \\
\frac{\sqrt{4A'x - 4A'C' + B'^2} - B'}{2A'} & a_i b_2 \leq x \leq a_i b_3 \\
\frac{\sqrt{4A'x - 4A'C' + B'^2} - B'}{2A'} & a_i b_3 < x < a_i b_4 \\
0 & x \geq a_i b_4
\end{cases}
\]

where, for Example 1, the membership function is:

\[
\mu_i(x) = \begin{cases} 
0 & x \leq 2 \\
\frac{\sqrt{16x + 49} - 9}{8} & 2 < x < 15 \\
1 & 15 \leq x \leq 30 \\
\frac{\sqrt{36x + 7} - 51}{18} & 30 < x < 72 \\
0 & x \geq 72
\end{cases}
\]

Next, consider two nonnegative normal fuzzy numbers \( \tilde{a} = (m_1, \sigma_1) \) and \( \tilde{b} = (m_2, \sigma_2) \). The end points of the \( \alpha \)-cuts for them are:

\[
\begin{align*}
\tilde{a}_L^\alpha &= m_1 - \sigma_1 \sqrt{-\ln \alpha} \\
\tilde{a}_R^\alpha &= m_1 + \sigma_1 \sqrt{-\ln \alpha} \\
\tilde{b}_L^\alpha &= m_2 - \sigma_2 \sqrt{-\ln \alpha} \\
\tilde{b}_R^\alpha &= m_2 + \sigma_2 \sqrt{-\ln \alpha}
\end{align*}
\]

Then,

\[
\tilde{c}_L^\alpha = \tilde{a}_L^\alpha \times \tilde{b}_L^\alpha = x = (m_1 - \sigma_1 \sqrt{-\ln \alpha})(m_2 - \sigma_2 \sqrt{-\ln \alpha}) = A(\sqrt{-\ln \alpha})^2 + B\sqrt{-\ln \alpha} + C,
\]

where, \( A = \sigma_1 \sigma_2 \), \( B = -(m_1 \sigma_2 + m_2 \sigma_1) \) and \( C = m_1 m_2 \).

Using (8), we have

\[
\alpha = e^{-\frac{\sqrt{4Ax - 4AC + B^2} - B}{2A}}. \tag{9}
\]

Also, for the right end of the \( \alpha \)-cuts, we obtain:

\[
\alpha = e^{-\frac{\sqrt{4Ax - 4AC + B^2} - B'}{2A}}. \tag{10}
\]

Using (9) and (10), the membership function for the product of the two nonnegative normal fuzzy numbers is:

\[
\mu_i(x) = \begin{cases} 
\frac{\sqrt{4Ax - 4AC + B^2} - B}{2A} & x \leq m_1 m_2 \\
\frac{\sqrt{4Ax - 4AC + B^2} - B'}{2A} & x > m_1 m_2
\end{cases}
\]

where, for Example 2, the membership function is:

\[
\mu_i(x) = \begin{cases} 
\frac{\sqrt{\frac{25600}{2x} - 560}}{72} & x \leq 2000 \\
\frac{\sqrt{\frac{25600}{2x} - 560}}{72} & x > 2000
\end{cases}
\]

We now show how to multiply a trapezoidal fuzzy number with a normal one. We present a numerical approach to construct a model for the product and its corresponding membership function.

Let \( \tilde{a} = (a_1, a_2, a_3, a_4) \) and \( \tilde{b} = (m, \sigma) \) be the trapezoidal and normal fuzzy numbers, respectively. Given \( \alpha_i \in (0, 1] \), the \( \alpha \)-product of these fuzzy numbers using equations (1) and (2) is obtained to be:

\[\tilde{a}_o \times \tilde{b}_o = [L, R],\]

where,
According to (11), using $\alpha_i = \alpha_{i+1} + \Delta \alpha_i$, $\alpha_0 = 0$, and $\Delta \alpha_i = \frac{1}{n}, i = 1, ..., 2n$ points are gained for $\tilde{c}$, $n$ points for the $\tilde{c}_{a^*}$ and $n$ points for the $\tilde{c}_{a^-}$. The membership function of the product is obtained using the resulting points via $\alpha$-cuts and a regression method for fitting a membership function to the product. We propose an exponential membership function for approximating the product as follows (later, we will see that this choice would indeed provide a good model for the approximating product of trapezoidal and normal fuzzy numbers). Let $x_i = \tilde{c}_{a^*}^R$ and $y_i = \mu(\tilde{c}_{a_i}^R)$, and for the $n$ points $(x_i, y_i)$, consider the fitting model to be $y = e^{-(x - \tilde{\lambda})^2}$. As defined, the unknown parameters $\lambda$ and $\beta$ appear nonlinearly.

To linearize the model, we note that for any $x_i > \lambda$, we must have

$$\ln y_i = -\frac{(x_i - \tilde{\lambda})^2}{\beta}.$$  

Since $0 < y_i \leq 1$, then $\ln(y_i) \leq 0$, and thus we can write

$$\sqrt{-\ln y_i} = \frac{(x_i - \lambda)}{\beta},$$  

and hence

$$\sqrt{-\ln y_i} + \lambda = x_i. \quad (14)$$

Therefore, the problem of least squares for the minimization of error is:

$$\min E = \sum_{i=1}^{n} (\beta \sqrt{-\ln y_i} + \lambda - x_i)^2, \quad (15)$$

where, $x_i = \tilde{c}_{a_i}^R$ is given by (11). To solve (15), we need to have:

$$\frac{\partial E}{\partial \beta} = \sum_i \left[ -2 \frac{1}{\beta} \sqrt{-\ln y_i} (x_i - \beta \sqrt{-\ln y_i} - \lambda) \right] = 0, \quad (16)$$

$$\frac{\partial E}{\partial \lambda} = \sum_i \left[ -2 (x_i - \beta \sqrt{-\ln y_i} - \lambda) \right] = 0,$$

and thereby, the following so-called normal equations are to be solved:

$$\beta \sum_i \ln y_i - \lambda \sum_i \sqrt{-\ln y_i} = -\sum_i x_i \sqrt{-\ln y_i}, \quad (17)$$

$$-\beta \sum_i \sqrt{-\ln y_i} - n \lambda = -\sum_i x_i. \quad (18)$$

From (17) and (18), $\lambda$ and $\beta$ are explicitly determined to be:

$$\lambda = \frac{-\sum_i x_i \sqrt{-\ln y_i} - \sum_i \sqrt{-\ln y_i}}{-\sum_i x_i - n}, \quad (19)$$
\[
\lambda = \frac{\sum \ln y_i - \sum \left(\ln y_i \times x_i\right)}{-\sum \sqrt{-\ln y_i} - \sum \left(-\ln y_i\right) + n} \Rightarrow \lambda = \frac{\sum \ln y_i - \sum \left(\ln y_i \times x_i\right) \times \left(\sum \sqrt{-\ln y_i}\right)}{-n \sum \sqrt{-\ln y_i} - \sum \sqrt{-\ln y_i} \times \sum \sqrt{-\ln y_i}}.
\]  

(20)

Now, similarly let \( x_i = \tilde{c}_i \) and \( y_i = \mu(\tilde{c}_i) \), and consider the model \( y = e^{-\frac{\lambda-x}{\beta}^2} \). Using the above approach, \( \lambda' \) and \( \beta' \) are found to be:

\[
\ln y_i = -\frac{\lambda' - x_i}{\beta'}, \\
\beta' = \frac{n \sum x_i \sqrt{-\ln y_i} - \sum x_i \times \sum \sqrt{-\ln y_i}}{n \sum \ln y_i + \sum \sqrt{-\ln y_i} \times \sum \sqrt{-\ln y_i}}, \\
\lambda' = \frac{\sum \ln y_i \times \sum x_i + \sum \left(x_i \times \sqrt{-\ln y_i}\right) \times \sum \sqrt{-\ln y_i}}{n \sum \ln y_i + \sum \sqrt{-\ln y_i} \times \sum \sqrt{-\ln y_i}}.
\]

(21)

(22)

(23)

Thus, the membership function is determined to be:

\[
\mu_c(x) = \begin{cases} 
    e^{-\frac{(x-\lambda)^2}{\beta}}, & x < \lambda' \\
    1, & \lambda' \leq x \leq \lambda \\
    e^{-\frac{(x-\lambda)^2}{\beta}}, & x > \lambda,
\end{cases}
\]

(24)

with \( \lambda, \beta, \lambda', \) and \( \beta' \) as defined by (19), (20), (22), and (23), respectively.

Next, we provide a numerical illustration of our proposed model for the product of trapezoidal and normal fuzzy numbers.

Example 3: Consider the following two fuzzy numbers, one being normal and the other being trapezoidal:

\( \tilde{a} = (5,9), \tilde{b} = (1,2,4,6) \).

For the normal fuzzy number, we consider \( y = e^{-\frac{(x-\lambda)^2}{\beta}} \), where \( \lambda \) and \( \beta \) are the mean and standard deviation, respectively. The diagrams of their product, \( \tilde{a} \times \tilde{b} \), using equation (11), is shown in figure 5, respectively. As appeared, the result is neither trapezoidal nor normal.

Using the points obtained from the a_1-cuts considering \( n = 100 \), the amounts of \((\lambda, \beta)\) for the right side and \((\lambda', \beta')\) for the left side are obtained by solving the corresponding least squares problems of the regression model.

To do this, we used the function CURVEFITTING in MATLAB. The obtained result is:

\[
\mu_c(x) = \begin{cases} 
    e^{-\frac{(x-\lambda)^2}{\beta}}, & x < \lambda' \\
    1, & \lambda' \leq x < \lambda \\
    e^{-\frac{(x-\lambda)^2}{\beta}}, & x > \lambda.
\end{cases}
\]
The obtained membership function using the least squares method and the product of fuzzy numbers using α-cuts is shown in figure 6.

5. Solving Fully Fuzzy Linear Systems

To conceptualize the solution method, consider the following $2 \times 2$ system,

$$\begin{bmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} \tilde{e} \\ \tilde{f} \end{bmatrix},$$

where, $\tilde{a}, \tilde{b}, ..., \tilde{f}$ are known fuzzy numbers of arbitrary types and $\tilde{x}, \tilde{y}$ are unknown fuzzy numbers (without loss of generality, all fuzzy numbers are considered to be nonnegative). For any $\alpha$-cuts, a corresponding system is configured as follows:

$$\begin{bmatrix} \tilde{a}^- & \tilde{a}^+ \\ \tilde{c}^- & \tilde{c}^+ \end{bmatrix} \begin{bmatrix} \tilde{x}^- & \tilde{x}^+ \\ \tilde{y}^- & \tilde{y}^+ \end{bmatrix} = \begin{bmatrix} \tilde{e}^- & \tilde{e}^+ \\ \tilde{f}^- & \tilde{f}^+ \end{bmatrix}.$$

Using the fuzzy addition and product, we have

$$\begin{bmatrix} \tilde{a}^- & \tilde{a}^+ \\ \tilde{c}^- & \tilde{c}^+ \end{bmatrix} \begin{bmatrix} \tilde{x}^- & \tilde{x}^+ \\ \tilde{y}^- & \tilde{y}^+ \end{bmatrix} = \tilde{e}^- \tilde{e}^+ \tilde{f}^- \tilde{f}^+.$$

Then, using (3), this system is turned into a $2 \times 2$ system with real numbers as follows:

$$\begin{align*}
\tilde{a}^- \tilde{x}^- + \tilde{b}^- \tilde{y}^- &= \tilde{e}^- \\
\tilde{a}^- \tilde{x}^+ + \tilde{b}^- \tilde{y}^+ &= \tilde{e}^+ \\
\tilde{c}^- \tilde{x}^- + \tilde{d}^- \tilde{y}^- &= \tilde{f}^- \\
\tilde{c}^- \tilde{x}^+ + \tilde{d}^- \tilde{y}^+ &= \tilde{f}^+.
\end{align*}$$

After solving these two systems for the $\alpha$-cuts, the four unknowns $\tilde{x}^-_a, \tilde{x}^+_a, \tilde{y}^-_a$ and $\tilde{y}^+_a$ are obtained. Next, using the regression the membership functions for $\tilde{x}$ and $\tilde{y}$ are obtained.

6. Conclusions

We proposed a practical approach for computing the product of fuzzy numbers using $\alpha$-cuts. To obtain the corresponding membership function for the resulting fuzzy product, we constructed a regression model by using a least squares model for the regression. The proposed model, while being practically simple, has the flexibility to construct the product operator on
various types of fuzzy numbers having the same type or being mixed. We made use of the approach to compute the solutions of fully fuzzy linear systems. The approach can also be used for computing solutions of fully fuzzy linear systems with various fuzzy variables.

References


